Ex 1)

$$\mathcal{L}(X \mid Y=0) = \mathcal{U}([0, \theta]), \quad \theta \in (0,1)$$
$$\mathcal{L}(X \mid Y=1) = \mathcal{U}([0,1])$$

$\Rightarrow x \in [0,1]$

$$p = \mathbb{P}(Y=1)$$

$$\eta(x) = \mathbb{P}(Y=1 \mid X=x).$$

D'après le cours, on a

$$\eta(x) = \frac{p f_+(x)}{p f_+(x) + (1-p) f_-(x)}$$ avec f_+ & f_- densités cond. aux classes.

$$= \frac{p \, \mathbb{1}_{[0,1]}(x)}{p \, \mathbb{1}_{[0,1]}(x) + (1-p) \frac{\mathbb{1}_{[0,\theta]}(x)}{\theta}}$$

- Si $\theta < x < 1$, $\eta(x) = p/p = 1$
- Si $1 < x \vee x < 0$ (indéfini) impossible pcq $x \in [0,1]$
- Si $0 \leq x \leq \theta$, $\eta(x) = \frac{p}{p + \frac{(1-p)}{\theta}} = \frac{\theta p}{\theta p + 1 - p}$

Donc, $\eta(x) = \begin{cases} 1 & \text{si } \theta < x \\ \dfrac{\theta p}{\theta p + 1 - p} & \text{si } 0 \leq x \leq \theta \end{cases}$

Si $\theta = 1/2$, $\eta(x) = \begin{cases} 1 & \frac{1}{2} < x \\ \dfrac{p/2}{p/2 + 1 - p} & 0 \leq x \leq \frac{1}{2} \end{cases}$

Ex 2

⚠ Dans le cours, $y \in \{-1, 1\}$!

$\mathcal{L}_X = P_X$, $\mathcal{L}_{Y|X=x} = \mathcal{B}er(\eta(x))$, $\eta(x) = \dfrac{x}{x+\theta}$,

$\forall \theta > 0$.

Par définition, $g^*(x) = \mathbb{1}_{\{\eta(x) > 1/2\}}$

Dans notre cas, $\eta(x) > \frac{1}{2} \iff \dfrac{x}{x+\theta} > \frac{1}{2}$

$\iff \dfrac{2x}{x+\theta} > 1 \iff x + \theta < 2x$

$\iff \theta < x$

Donc, $g^*(x) = \mathbb{1}_{\{x > \theta\}}$.

Par définition, $L(g^*) = L^* = \mathbb{E}\left[\min\{\eta(x) ; 1-\eta(x)\}\right]$

$$= \mathbb{E}\left[\mathbb{1}_{\{x \leq \theta\}}\eta(x) + \mathbb{1}_{\{x > \theta\}}(1-\eta(x))\right]$$

Si $x \leq \theta$, alors $\eta(x) \leq \frac{1}{2}$
donc $\min\{\eta(x) ; 1-\eta(x)\} = \eta(x)$.

On a la densité de la distribution marginale P_X :

$$P_X : \mathbb{R} \to \mathbb{R}^+$$
$$x \mapsto \frac{\mathbb{1}_{[0, \alpha\theta]}}{\alpha\theta}, \quad \forall \alpha > 1.$$

On peut donc réécrire l'espérance :

$$L^* = \int_{\mathbb{R}} \mathbb{1}_{\{x \leq \theta\}}\eta(x) + \mathbb{1}_{\{x > \theta\}}(1-\eta(x))\, d\mathbb{P}(x)$$

$$= \int_{\mathbb{R}} \left[\mathbb{1}_{\{x \leq \theta\}}\eta(x) + \mathbb{1}_{\{x > \theta\}}(1-\eta(x))\right] P_X(x)\, dx$$

$$= \frac{1}{\alpha\theta} \int_0^{\alpha\theta} \mathbb{1}_{\{x \leq \theta\}}\eta(x) + \mathbb{1}_{\{x > \theta\}}(1-\eta(x))\, dx$$

$$L^* = \frac{1}{\alpha\theta}\left(\int_0^{\theta} \eta(x)\,dx + \int_{\theta}^{\alpha\theta} 1-\eta(x)\,dx\right)$$

$$= \frac{1}{\alpha\theta}\left(\int_0^{\theta} 1-\left(1-\eta(x)\right)dx + \int_{\theta}^{\alpha\theta} 1-\eta(x)\,dx\right)$$

$$= \frac{1}{\alpha\theta}\left(\int_0^{\theta} 1-\frac{\theta}{x+\theta}\,dx + \int_{\theta}^{\alpha\theta} \frac{\theta}{x+\theta}\,dx\right)$$

$$= \frac{1}{\alpha\theta}\left(\left[x-\theta\ln(x+\theta)\right]_0^{\theta} + \left[\theta\ln(x+\theta)\right]_{\theta}^{\alpha\theta}\right)$$

$$= \frac{1}{\alpha\theta}\left(\theta-\theta\ln(2\theta)+\theta\ln(\theta)+\theta\ln(\alpha\theta+\theta)-\theta\ln(2\theta)\right)$$

$$= \frac{1}{\alpha\theta}\left(\theta-2\theta\ln(2\theta)+\theta\ln\theta+\theta\ln\left(\theta(\alpha+1)\right)\right)$$

$$= \frac{1}{\alpha\cancel{\theta}}\left(\theta\left(1-2\ln(2\theta)+2\ln\theta+\ln(\alpha+1)\right)\right)$$

$$= \frac{1}{\alpha}\left(1+2\ln\left(\frac{\theta}{2\theta}\right)+\ln(\alpha+1)\right)$$

$$= \frac{1}{\alpha}\left(1-2\ln 2+\ln(\alpha+1)\right)$$

On peut calculer la dérivée :

$$\frac{\partial L^*}{\partial \alpha} = \frac{1}{\alpha}\frac{1}{\alpha+1} - \left(1 - 2\ln 2 + \ln \alpha+1\right)\frac{1}{\alpha^2}$$

$$= \frac{1}{\alpha^2}\left(2\ln 2 - \ln \alpha+1 - 1\right) + \frac{1}{\alpha}\frac{1}{\alpha+1}$$

Si on développe, on voit que

$$\circledast \quad \frac{\partial L^*}{\partial \alpha} > 0 \iff 4 > (\alpha+1)e^{1/\alpha+1}$$

Si $\alpha = 1$, $\frac{\partial L^*}{\partial \alpha} > 0$ et si $\alpha = 2$, $\frac{\partial L^*}{\partial \alpha} < 0$. Comme $(\alpha+1)e^{1/\alpha+1}$ est strictement croissant sur $[1, +\infty[$, $\alpha^* \in]1, 2[$.

On peut trouver numériquement $\alpha \approx 1,80$

On peut même se rapporter à $\frac{2^{\alpha+1}}{e} > (\alpha+1)^{\alpha+1}$ (aucune solution analytique). $\circledast$

$$\frac{\partial L^*}{\partial \alpha} = \frac{1}{\alpha^2}\left(2\ln 2 - \ln_{\alpha+1} - 1\right) + \frac{1}{\alpha}\frac{1}{\alpha+1}$$

$$= \frac{2\ln 2 - \ln_{\alpha+1} - 1}{\alpha^2} + \frac{1}{\alpha^2+\alpha}$$

$$= \frac{(2\ln 2 - \ln_{\alpha+1} - 1)(\alpha^2+\alpha)}{\alpha^2(\alpha^2+\alpha)} + \frac{\alpha^2}{\alpha^2(\alpha^2+\alpha)}$$

$$= \frac{1}{\alpha^2(\alpha^2+\alpha)}\left(\begin{array}{l} 2\alpha^2\ln 2 - \alpha^2 \ln_{\alpha+1} - \cancel{\alpha^2} + 2\alpha\ln(2) \\ -\alpha\ln_{\alpha+1} - \alpha + \cancel{\alpha^2} \end{array}\right)$$

$$= \frac{\alpha}{\alpha^2(\alpha^2+\alpha)}\left(2\alpha\ln 2 - \alpha\ln_{\alpha+1} + 2\ln 2 - \ln_{\alpha+1} - 1\right) > 0$$

$$\Leftrightarrow 2\alpha\ln 2 + 2\ln 2 > \alpha\ln_{\alpha+1} + \ln_{\alpha+1} + 1$$

$$\Leftrightarrow (2\alpha+2)\ln 2 > (\alpha+1)\ln(\alpha+1) + 1$$

$$\Leftrightarrow \ln\left(2^{2\alpha+2}\right) > \ln\left((\alpha+1)^{\alpha+1}\right) + \ln e$$

$$\Leftrightarrow \ln\left(2^{2\alpha+2}\right) > \ln\left((\alpha+1)^{\alpha+1} e\right)$$

$$\Leftrightarrow 2^{2\alpha+2} > (\alpha+1)^{\alpha+1} e$$

$$\Leftrightarrow \frac{2^{2\alpha+2}}{e} > (\alpha+1)^{\alpha+1}$$

$$\ln 2 > \frac{1}{2}\ln(\alpha+1) + \frac{1}{2(\alpha+1)}$$

$$\ln 2 > \frac{1}{2}\left(\ln(\alpha+1) + \frac{1}{\alpha+1}\right)$$

$$\ln 2 > \frac{1}{2}\left(\ln(\alpha+1)e^{1/(\alpha+1)}\right)$$

$$\ln 4 > \ln\left(\alpha+1\, e^{1/(\alpha+1)}\right)$$

$$4 > (\alpha+1)e^{1/(\alpha+1)}$$

Ex 3

$X = (T, U, V)$ où T, U, V iid et $T, U, V \sim \mathcal{E}(1)$.

$Y = \mathbb{1}_{\{T+U+V<\theta\}}$ $\forall \theta > 0$.

$\mathcal{L}_X = \mathcal{E}(1)^{\otimes}$

1) V n'est pas observé

$$\eta(t,u) = \mathbb{P}(T+U+V<\theta \mid T=t, U=u)$$

$$= \mathbb{P}(V<\theta-T-U \mid T=t, U=u)$$

On a

$$\eta(t,u) = \mathbb{E}_V\left[\mathbb{1}_{\{V<\theta-t-u\}}\right]$$

$$= \int_0^{\theta-t-u} \underbrace{\mathbb{1}_{\{t+u<\theta\}}}_{t+u>\theta \Leftrightarrow \theta-T-U<0 \Leftrightarrow \mathbb{E}_V[\dots]=0} e^{-v}\, dv$$

$$= \mathbb{1}_{\{t+u<\theta\}} \left[-e^{-v}\right]_0^{\theta-t-u} = \mathbb{1}_{\{t+u<\theta\}} \left(1-e^{-(\theta-t-u)}\right)$$

On calcule le classifieur de Bayes :

$$g^*(t,u) = \mathbb{1}\{\eta(t,u) > 1/2\}$$

$$\eta(t,u) > \frac{1}{2} \iff \begin{cases} t+u \leq \theta \\ e^{-(\theta-t-u)} < \frac{1}{2} \end{cases} \iff \begin{cases} t+u \leq \theta \\ \theta-t-u > \ln 2 \end{cases}$$

$$\iff \theta > t+u+\ln 2$$

$$g^*(t,u) = \mathbb{1}\{t+u < \theta - \ln 2\}$$

On peut calculer l'erreur de Bayes :

$$L^* = \mathbb{E}_{t,u}\left[\min\{\eta(t,u) ; 1-\eta(t,u)\}\right]$$

$$= \mathbb{E}_{t,u}\left[\mathbb{1}_{\{t+u \geq \theta-\ln 2\}}\eta(t,u) + \mathbb{1}_{\{t+u \leq \theta-\ln 2\}}(1-\eta(t,u))\right]$$

Calculons la densité de $Z = T+U$:

$$f_Z(z) = \int_{-\infty}^{+\infty} f_T(u)\, f_U(z-u)\, du$$

$$f_Z(z) = \int_{-\infty}^{+\infty} \mathbb{1}_{\mathbb{R}_+}(k)\, \mathbb{1}_{\mathbb{R}_+}(z-k)\, e^{-k} e^{-(z-k)}\, dk$$

$$= \int_0^z e^{-k} e^{-(z-k)}\, dk$$

$$= \int_0^z e^{-z}\, dk = e^{-z} \lambda([0,z]) = z e^{-z}.$$

On peut donc ré-écrire l'espérance :

$$L^* = \mathbb{E}_Z\left[\mathbb{1}_{\{Z \geq \theta - \ln 2\}}\, \eta(Z) + \mathbb{1}_{\{Z < \theta - \ln 2\}} (1 - \eta(Z)) \right]$$

avec $\eta(z) = \left(1 - e^{-\theta + z}\right) \mathbb{1}_{\{z < \theta\}}$.

$$L^* = \int_0^{\theta - \ln 2} (1 - \eta(z))\, f_Z(z)\, dz + \int_{\theta - \ln 2}^{\theta} \eta(z)\, f_Z(z)\, dz$$

$$= \int_0^{\theta - \ln 2} e^{-\theta + z}\, z e^{-z}\, dz + \int_{\theta - \ln 2}^{\theta} \left(1 - e^{-\theta + z}\right) z e^{-z}\, dz$$

$$= \int_0^{\theta - \ln 2} e^{-\theta} z\, dz + \int_{\theta - \ln 2}^{\theta} z e^{-z} - z e^{-\theta}\, dz$$

$$= e^{-\theta} \left[\tfrac{1}{2} z^2\right]_0^{\theta - \ln 2} + \boxed{\left[-z e^{-z}\right]_{\theta - \ln 2}^{\theta} + \left[-e^{-z}\right]_{\theta - \ln 2}^{\theta}} - e^{-\theta} \left[\tfrac{1}{2} z^2\right]_{\theta - \ln 2}^{\theta}$$

$$\int z e^{-z} = \int_a^b z\,(-e^{-z})'\,dz$$

$$= \left[-z e^{-z}\right]_a^b - \int_a^b -e^{-z}\,dz$$

$$= \left[-z e^{-z}\right]_a^b + \left[-e^{-z}\right]_a^b$$

$$L^* = e^{-\theta}\frac{(\theta-\ln 2)^2}{2} - \theta e^{-\theta} + (\theta - \ln 2)e^{-(\theta-\ln 2)} - e^{-\theta} + e^{-(\theta-\ln 2)}$$

$$- \frac{\theta^2}{2}e^{-\theta} + \frac{(\theta-\ln 2)^2}{2}e^{-\theta}$$

$$= -e^{-\theta}\left(1 + \theta + \frac{\theta^2}{2} - (\theta-\ln 2)^2\right) + e^{-(\theta-\ln 2)}\left(1 + (\theta - \ln 2)\right)$$

$$= -e^{-\theta}\left(1 + \theta + \frac{\theta^2}{2} - (\theta-\ln 2)^2\right) + 2e^{-\theta}\left(1 + \theta - \ln 2\right)$$

$$= e^{-\theta}\left(-1 - \theta - \frac{\theta^2}{2} + (\theta-\ln 2)^2 + 2 + 2\theta - 2\ln 2\right)$$

$$= e^{-\theta}\left(-1 - \theta - \frac{\theta^2}{2} + \theta^2 - 2\theta\ln 2 + \ln^2 2 + 2 + 2\theta - 2\ln 2\right)$$

$$= e^{-\theta}\left(1 + \theta + \frac{\theta^2}{2} - 2\ln 2\,(1+\theta) + \ln^2 2\right)$$

Si $\theta = 9$, $L^* \approx 0,0046$.

2) Seulement T est observé.

$$q(t) = \mathbb{P}\left(T+U+V < \theta \mid T\right)$$

$$= \mathbb{P}\left(U+V < \theta - T \mid T\right)$$

On pose $Z = U+V$.

$$q(t) = \mathbb{1}_{\{t<\theta\}} \int_0^{\theta - t} z e^{-z} \, dz = \mathbb{1}_{\{t<\theta\}} \left(\left[-z e^{-z} \right]_0^{\theta-t} - \left[e^{-z} \right]_0^{\theta - t} \right)$$

$$= \mathbb{1}_{\{t<\theta\}} \left(-(\theta - t) e^{-(\theta - t)} - e^{-(\theta-t)} + 1 \right)$$

$$= \mathbb{1}_{\{t<\theta\}} \left(1 + \left(-(\theta - t) - 1 \right) e^{-(\theta - t)} \right)$$

$$= \mathbb{1}_{\{t<\theta\}} \left(1 - (\theta - t + 1) e^{-(\theta - t)} \right)$$

$$q(t) > \frac{1}{2} \iff \begin{cases} t < \theta \\ 1 - (\theta - t + 1) e^{-(\theta - t)} > 1/2 \end{cases}$$

$$\iff \begin{cases} t < \theta \\ (\theta - t + 1) e^{-(\theta - t)} < 1/2 \end{cases}$$

$$\Leftrightarrow e^{-(\theta-t)} < \frac{1}{2(\theta-t+1)} \quad \wedge \quad t<\theta$$

$$\Leftrightarrow t-\theta < -\log\left(2(\theta-t+1)\right) \quad \wedge \quad t\,\theta$$

$$\Leftrightarrow t-\theta < -\log 2 - \log(\theta-t+1) \quad \wedge \quad t<\theta$$

$$\Leftrightarrow \theta-t-\log(\theta-t+1) > \log 2 \quad \wedge \quad t<\theta$$

$$\eta(t) > \frac{1}{2} \Leftrightarrow \theta-t-\log(\theta-t+1) > \log 2 \wedge t<\theta$$

$$g^*(t) = \mathbb{1}_{\{\theta-t-\log(\theta-t+1)>\log 2\}} \mathbb{1}_{\{t<\theta\}}$$

$$L^* = \mathbb{E}\left[\min\{\eta(t) ; 1-\eta(t)\}\right]$$

$$= \mathbb{E}\left[\mathbb{1}_{\{\theta-t-\log(\theta-t+1)<\log 2\}} \eta(t) + \right.$$

$\mathbb{1}_{\{t<\theta\}}$

$$\left. \mathbb{1}_{\{\theta-t-\log(\theta-t+1)>\ln 2\}} \mathbb{1}_{\{t<\theta\}} \left(1-\eta(t)\right)\right]$$

$$= \int_0^\theta \left[\mathbb{1}_{\{\dots\}}\, 1-(\theta-t+1)e^{-(\theta-t)} + \mathbb{1}_{\{\dots\}}(\theta-t+1)e^{-(\theta-t)}\right] e^{-t}\, dt$$

Si $\theta = 9$

$L^* \approx 0{,}0059$

Plus grand que lorsque T et U sont observés.

3) Aucun n'est observé.

On pose $Z = T + U$ et $S = V + Z$. On calcule la densité de S, $f_S(s) = \int_0^{+\infty} z e^{-z} e^{-(s-z)} \underbrace{\mathbb{1}_{\mathbb{R}_+}(s-z)}_{V \in \mathbb{R}_+} dz$

$$= \int_0^s z e^{-s} dz = e^{-s} \left[\frac{z^2}{2} \right]_0^s = \frac{s^2 e^{-s}}{2}$$

On calcule $\eta(x) = \mathbb{P}(Y = 1 \mid X = x) = \mathbb{P}(S < \theta)$

$$= \int_0^\theta \frac{s^2 e^{-s}}{2} ds$$

$$= \left[\frac{-s^2 + 2s + 2}{2} e^{-s} \right]_0^\theta = 1 - \frac{e^{-\theta}}{2}(\theta^2 + 2\theta + 2)$$

$$\eta(x) \geq \frac{1}{2} \iff e^{-\theta}(\theta^2 + 2\theta + 2) < 1$$

$$g^* = \mathbb{1}\{e^{-\theta}(\theta^2 + 2\theta + 2) < 1\}$$

$$L^* = \mathbb{E}[\min\{\eta(x) ; 1 - \eta(x)\}]$$

$$= \mathbb{E}\left[\mathbb{1}\{e^{-\theta}(\theta^2 + 2\theta + 2) \geq 1\} \eta(x) + \mathbb{1}_{\{\dots\}}(1 - \eta(x))\right]$$

Si $\theta = 9$,

$L^* \approx 0,0062$ Encore plus grand que 9° 2 !

Ex 4

1) $$f^{+}(x) = \frac{1}{(2\pi)^{d/2} |\Sigma_+|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_+)^T \Sigma_+^{-1} (x-\mu_+)\right)$$

$$f^{-}(x) = \frac{1}{(2\pi)^{d/2} |\Sigma_-|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_-)^T \Sigma_-^{-1} (x-\mu_-)\right)$$

$$\eta = \frac{p f_+}{p f_+ + (1-p) f_-}$$

Donc, $$\eta(x) = \frac{\frac{p}{(2\pi)^{d/2} |\Sigma_+|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_+)^T \Sigma_+^{-1} (x-\mu_+)\right)}{\frac{p}{(2\pi)^{d/2} |\Sigma_+|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_+)^T \Sigma_+^{-1} (x-\mu_+)\right) + \frac{1-p}{(2\pi)^{d/2} |\Sigma_-|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu_-)^T \Sigma_-^{-1} (x-\mu_-)\right)}$$

On pose $\pi_{\{+,-\}} = (2\pi)^{d/2} |\Sigma_{\{+,-\}}|^{1/2}$ et

$$e_{\{+,-\}} = \exp\left(-\frac{1}{2}\left((x-\mu_{\{+,-\}})^T \Sigma_{\{+,-\}}^{-1} (x-\mu_{\{+,-\}})\right)\right)$$

Étudions $\frac{p e_+}{\pi_+} + \frac{(1-p) e_-}{\pi_-}$:

$$\frac{\pi_- p\, e_+}{\pi_+ \pi_-} + \frac{(1-p)\pi_+ e_-}{\pi_+ \pi_-} = \frac{\pi_- p\, e_+ + (1-p)\pi_+ e_-}{\pi_+ \pi_-}$$

$$= \frac{p\pi_- e_+ + \pi_+ e_- - p\pi_+ e_-}{\pi_+ \pi_-}$$

$$= \frac{p\pi_- e_+ + (1-p)\pi_+ e_-}{\pi_+ \pi_-}$$

$$\eta(x) = \frac{p e_+}{\cancel{\pi_+}} \frac{\cancel{\pi_+}\pi_-}{p\pi_- e_+ + (1-p)\pi_+ e_-} = \frac{\cancel{p} e_+ \pi_-}{\cancel{p}\left(\pi_- e_+ + \left(\frac{1-p}{p}\right)\pi_+ e_-\right)}$$

$$= \frac{e_+ \pi_-}{\pi_- e_+ + \left(\frac{1-p}{p}\right)\pi_+ e_-}$$

$$= \frac{e_+ \pi_-}{e_+ \pi_-\left(1 + \frac{1-p}{p}\frac{\pi_+ e_-}{\pi_- e_+}\right)}$$

$$= \frac{1}{1 + \frac{1-p}{p}\frac{(2\pi)^{d/2}|\Sigma_+|^{1/2}}{(2\pi)^{d/2}|\Sigma_-|^{1/2}}\frac{\exp\left(-\frac{1}{2}(x-\mu_-)^T\Sigma_-^{-1}(x-\mu_-)\right)}{\exp\left(-\frac{1}{2}(x-\mu_+)^T\Sigma_+^{-1}(x-\mu_+)\right)}}$$

$$= \frac{1}{1 + \frac{1-p}{p}\left|\frac{\Sigma_+}{\Sigma_-}\right|^{1/2}\exp\left(-\frac{1}{2}\left((x-\mu_-)^T\Sigma_-^{-1}(x-\mu_-) - (x-\mu_+)^T\Sigma_+^{-1}(x-\mu_+)\right)\right)}$$

2) $$\log\left(\frac{\eta_\theta(x)}{1-\eta_\theta(x)}\right) = \theta^T x$$

$$\Rightarrow \frac{\eta_\theta(x)}{1-\eta_\theta(x)} = e^{\theta^T x}$$

On peut donc écrire :

$$\eta_\theta(x)\left(1+e^{\theta^T x}\right) = \eta_\theta(x) + \frac{\eta_\theta^2(x)}{1-\eta_\theta(x)}$$

$$= \frac{\eta_\theta(x) - \eta_\theta^2(x) + \eta_\theta^2(x)}{1-\eta_\theta(x)} = \frac{\eta_\theta(x)}{1-\eta_\theta(x)} = e^{\theta^T x}$$

On obtient :

$$\eta_\theta(x) = \frac{e^{\theta^T x}}{1+e^{\theta^T x}} = \frac{1}{1+e^{-\theta^T x}}.$$

Par définition, on sait que $\eta_\theta(x) = \frac{p f_+}{f_x}$,

avec $f_x = p f_+ + (1-p) f_-$.

On a donc $\begin{cases} f_+ = 1/p \\ f_- = \frac{f_x - p f_+}{1-p} = \frac{1+e^{-\theta^T x} - 1}{1-p} = \frac{e^{-\theta^T x}}{1-p} \end{cases}$

Ex 5)

1) $w \in]0,1[$.

$$L_w(g) = 2\mathbb{E}_{X,Y}\Big[(1-w)\,\mathbb{1}_{\{Y=1\}}\mathbb{1}_{\{g(X)=-1\}} + w\,\mathbb{1}_{\{Y=-1\}}\mathbb{1}_{\{g(X)=1\}}\Big]$$

$$= \mathbb{E}_{X,Y}\Big[(1-w)\underbrace{\mathbb{E}_Y\big[\mathbb{1}_{\{Y=1\}}\,|\,X\big]}_{\text{Law of total expectation}}\mathbb{1}_{\{g(X)=-1\}} + w\,\mathbb{E}_Y\big[\mathbb{1}_{\{Y=-1\}}\,|\,X\big]\mathbb{1}_{\{g(X)=1\}}\Big]$$

$$= \mathbb{E}\Big[(1-w)\,\eta(X)\,\mathbb{1}_{\{g(X)=-1\}} + w\,(1-\eta(X))\,\mathbb{1}_{\{g(X)=1\}}\Big]$$

Donc, $g^*(X) = 1 \iff w(1-\eta(X)) \leq (1-w)\,\eta(X)$

(g^* minimise L. Donc, si $g^*(X)=1$, $w(1-\eta(X))$ est plus petit que $(1-w)\,\eta(X)$)

$$\iff w - w\eta(X) \leq \eta(X) - w\,\eta(X)$$

$$\iff w \leq \eta(X)$$

Donc, $g^*(X) = 2\,\mathbb{1}_{\{w \leq \eta(X)\}} - 1$.

Calculons L^* :

$$L_w(g^*) = 2\mathbb{E}\left[\mathbb{1}_{\{\eta(X)<w\}}(1-w)\eta(X) + \mathbb{1}_{\{w\leq\eta(X)\}}w(1-\eta(X))\right]$$

$$= 2\mathbb{E}\left[\left(\mathbb{1}_{\{\eta(X)<w\}} + \mathbb{1}_{\{w\leq\eta(X)\}}\right)\min\left\{(1-w)\eta(X) ; w(1-\eta(X))\right\}\right]$$

$$= 2\mathbb{E}\left[\min\left\{(1-w)\eta(X) ; w(1-\eta(X))\right\}\right]$$

2) $u \in]0,1[$

On cherche g tq. $g \in \underset{g}{\operatorname{argmin}}\, \mathbb{P}(Y \neq g(X))$

s.t. $\mathbb{P}(g(X)=1) = u$.

On peut trouver un q tq $\mathbb{P}(\eta(X) > q) = u$.

Posons $q = q_{1-u}(\eta(X))$ $\left(\mathbb{P}(\eta(X) \leq q) = 1-u \iff 1 - \mathbb{P}(\eta(X)\leq q) = u \iff \mathbb{P}(\eta(X)>q) = u \right)$

On a $L_u(g) = \mathbb{E}\left[\mathbb{1}_{\{g(X)=-1\}}\eta(X) + \mathbb{1}_{\{g(X)=1\}}(1-\eta(X))\right]$

$$= \mathbb{E}\left[(\eta(X) - q)\, \mathbb{1}_{\{g(X)=-1\}} + q\, \mathbb{1}_{\{g(X)=-1\}} + (q - \eta(X))\, \mathbb{1}_{\{g(X)=1\}} + (1-q)\, \mathbb{1}_{\{g(X)=1\}} \right]$$

Or,

- $\mathbb{E}\left[q\, \mathbb{1}_{\{g(X)=-1\}} \right] = q\, \mathbb{P}(g(X) \neq 1) = q(1-u)$

- $\mathbb{E}\left[(1-q)\, \mathbb{1}_{\{g(X)=1\}} \right] = (1-q)\, \mathbb{P}(g(X)=1) = (1-q)u$

On peut donc écrire

$$L_u(g) = \mathbb{E}\left[(\eta(X)-q)\, \mathbb{1}_{\{g(X)=-1\}} + (q-\eta(X))\, \mathbb{1}_{\{g(X)=1\}} \right] + q(1-u) + (1-q)u.$$

Par définition, $g^*(X) = 1 \iff q - \eta(X) < \eta(X) - q$

$$\iff -2\eta(X) < -2q$$

$$\iff q < \eta(X)$$

Donc, $g^*(X) = 2\, \mathbb{1}_{\{q < \eta(X)\}} - 1$ et, $\mathbb{P}(g^*(X)=1) = \mathbb{P}(\eta(X) > q) = u$.

$$L_u^* = \mathbb{E}\left[\min\{ \eta(X) - q \,;\, q - \eta(X) \} \right] + q(1-u) + (1-q)u.$$

3) $\textcircled{R}$ est une valeur spécifique de rejet.

$\gamma \in \left]0, \frac{1}{2}\right[$.

$$L_\gamma^{R}(g) = \mathbb{P}(Y \neq g(X), g(X) \neq \textcircled{R}) + \gamma\, \mathbb{P}(g(X) = \textcircled{R})$$

$$= \mathbb{E}\left[\mathbb{1}_{\{g(X)=-1\}}\, \eta(X) + \mathbb{1}_{\{g(X)=1\}}(1-\eta(X))\right] + \gamma\, \mathbb{P}(g(X) = \textcircled{R})$$

$$= \mathbb{E}\left[\mathbb{1}_{\{g(X)=-1\}}\, \eta(X) + \mathbb{1}_{\{g(X)=1\}}(1-\eta(X)) + \gamma\, \mathbb{1}_{\{g(X)=\textcircled{R}\}}\right]$$

Si $g^*(X) = 1 \iff 1-\eta(X) < \eta(X) \wedge 1-\eta(X) < \gamma$

$g^*(X) = -1 \iff \eta(X) < 1-\eta(X) \wedge \eta(X) < \gamma$

$g^*(X) = \textcircled{R} \iff \gamma < \eta(X) \wedge \gamma < 1-\eta(X)$

Donc

$$g^*(X) = \begin{cases} 1 & \text{si } 1-\eta(X) < \eta(X) \wedge 1-\eta(X) < \gamma \\ -1 & \text{si } \eta(X) < 1-\eta(X) \wedge \eta(X) < \gamma \\ \textcircled{R} & \text{si } \gamma < \eta(X) \wedge \gamma < 1-\eta(X) \end{cases}$$

$\forall \gamma \in \left]0, \frac{1}{2}\right[$

$\gamma \leq 1-\gamma$.

Donc $\eta(X) < \gamma$

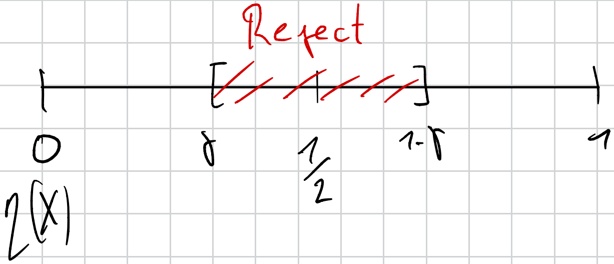
$\Rightarrow 1-\eta(X) > 1-\gamma > \gamma > \eta(X)$

et $1-\eta(X) < \gamma$

$\iff -\eta(X) < \gamma - 1$

$\Rightarrow \eta(X) > 1-\gamma > \gamma > 1-\eta(X)$

$$= \begin{cases} 1 & \text{si } 1-\eta(X) < \gamma \\ -1 & \text{si } \eta(X) < \gamma \\ \textcircled{R} & \text{si } \gamma < \eta(X) < 1-\gamma \end{cases}$$


Reject
0
γ
1/2
1-γ
1
λ(X)

Ex 8

5) $A(f) = \mathbb{E}[\max\{0, 1 - Y f(X)\}]$

$$= \mathbb{E}\left[\max\{0, 1 - f(X)\}\mathbb{1}_{\{Y=1\}} + \max\{0, 1 + f(X)\}\mathbb{1}_{\{Y=-1\}}\right]$$

$$= \mathbb{E}\left[\max\{0, 1 - f(X)\}\mathbb{E}[\mathbb{1}_{\{Y=1\}} | X] + \max\{0, 1 + f(X)\}\mathbb{E}[\mathbb{1}_{\{Y=-1\}} | X]\right]$$

$$= \mathbb{E}\left[\max\{0, 1 - f(X)\}\eta(X) + \max\{0, 1 + f(X)\}(1 - \eta(X))\right]$$

$$= \mathbb{E}\left[\mathbb{1}_{\{f(X) \leq 1\}}(1 - f(X))\eta(X) + \mathbb{1}_{\{f(X) \geq -1\}}(1 + f(X))(1 - \eta(X))\right]$$

$$= \mathbb{E}\Big[\mathbb{1}_{\{f(X) < -1\}}(1 - f(X))\eta(X) \quad > 2\eta(X) \text{ si: } f(X) < -1$$

$$2\min\{\eta(X), 1-\eta(X)\} \leq + \mathbb{1}_{\{-1 \leq f(X) \leq 1\}}(1 - f(X))\eta(X) + (1 + f(X))(1 - \eta(X)) \leq 2\max\{\eta(X), 1-\eta(X)\}$$

si: $f(X) \in [-1, 1]$.

$$+ \underbrace{\mathbb{1}_{\{f(X) > 1\}}(1 + f(X))(1 - \eta(X))}_{> 2(1-\eta(X)) \text{ si: } f(X) > 1}\Big]$$

$$f(X) = 2\mathbb{1}_{\{1-\eta(X) < \eta(X)\}} - 1$$

$$= 2\mathbb{1}_{\{\frac{1}{2} < \eta(X)\}}$$

Disjonction de cas :

Fixons X.

Si $\eta(X) < 1 - \eta(X)$:

si $f(X) < -1$:

$$(1 - f(X))\eta(X) > 2\eta(X)$$

si $f(X) > 1$:

$$(1 + f(x))(1 - \eta(X)) > 2(1 - \eta(X)) > 2\eta(X)$$

si $f(X) \in [-1, 1]$:

$$2\eta(X) \leq (1 - f(X))\eta(X) + (1 + f(X))(1 - \eta(X)) \leq 2(1 - \eta(X))$$

$2\eta(X)$ minore donc ces 3 termes. $f(X) = -1$ minimise donc l'intégrande pour un X fixé si $\eta(X) < 1 - \eta(X)$.

De la même manière, si $1 - \eta(X) < \eta(X)$, $2(1 - \eta(X))$ minore les 3 termes. $f(X) = -1$ minimise donc l'intégrande pour un X fixé si $1 - \eta(X) \leq \eta(X)$.

Donc, $f(X) = 2\,\mathbb{1}_{\{\eta(X) > \frac{1}{2}\}} - 1$.